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Abstract. This paper is a report on a computer check of some important positivity properties of the Hecke algebra in type H_4, including the non-negativity of the structure constants in the Kazhdan-Lusztig basis. This answers a long-standing question of Lusztig's. The same algorithm, carried out by hand, also allows us to deal with the case of dihedral Coxeter groups.

1 Statement of the problem

1.1. Let W be a Coxeter group, S its set of distinguished generators, and denote $\leq$ the Bruhat ordering on W. Denote $\mathcal{H}$ the Hecke algebra of W over the ring of Laurent polynomials $A = \mathbf{Z}[v, v^{-1}]$, where v is an indeterminate. We refer to [3] for basic results about Coxeter groups and Hecke algebras; we just recall here that $\mathcal{H}$ is a free A-module with basis $(t_y)_{y\in W}$, where the algebra structure is the unique one which satisfies

$$t_s.t_y = \begin{cases} t_{sy} & \text{if } sy > y \\ (v - v^{-1})t_y + t_{sy} & \text{if } sy < y \end{cases}$$

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(here we use $t_y = v^{-l(y)}T_y$, where the T_y satisfy the more familiar relation $T_s.T_y = (q-1)T_y + qT_{sy}$ when $sy < y$, with $q = v^2$.) Then there is a unique ring involution i on $\mathcal{H}$ such that $i(v) = v^{-1}$, and $i(t_y) = t_{y^{-1}}^{-1}$.

The Kazhdan-Lusztig basis of $\mathcal{H}$ is the unique family $(c_y)_{y \in W}$ such that *(a)* $i(c_y) = c_y$ and *(b)* $c_y = t_y + \sum_{x<y} p_{x,y} t_x$, with $p_{x,y} \in v^{-1}\mathbf{Z}[v^{-1}]$; in particular we find that $c_s = t_s + v^{-1}$ for all $s \in S$. It turns out that $P_{x,y} = v^{l(y)-l(x)} p_{x,y}$ is a polynomial in q, the *Kazhdan-Lusztig polynomial* for the pair x, y. For any pair (x, y) of elements in W, write

$$c_x.c_y = \sum_{z \in W} h_{x,y,z} c_z$$

(in other words, the $h_{x,y,z} \in A$ are the structure constants of the Hecke algebra in the Kazhdan-Lusztig basis).

A number of the deeper results in the theory of Hecke algebras depend on positivity properties of the polynomials $P_{x,y}$ and $h_{x,y,z}$. More precisely, consider the following properties :

P_1: the polynomials $P_{x,y}$ have non-negative coefficents;

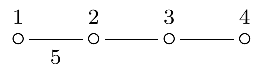
P_2: the $P_{x,y}$ are decreasing for fixed y, in the sense that if $x \leq z \leq y$ in W, the polynomial $P_{x,y} - P_{z,y}$ has non-negative coefficents;

P_3: the polynomials $h_{x,y,z}$ have non-negative coefficients.

Properties P_1 and P_3 are basic tools in the study of Kazhdan-Lusztig cells and the asymptotic Hecke algebra; they have been proved in [6] and [9] for crystallographic W using deep properties of intersection cohomology (see the remarks in section 3 of Lusztig [7] for the case where W is infinite.) Property P_2 is proved for finite Weyl groups in [4], using the description of Kazhdan-Lusztig polynomials in terms of filtrations of Verma modules.

None of these geometric or representation-theoretic interpretations are available in the non-crystallographic case, and the validity in general of the positivity properties above are among the main open problems in the theory of Coxeter groups. Let us concentrate on the case where W is finite. It is easy to see that for the validity of P_1–P_3 we may reduce to the case where W is irreducible.

1.2. The case where W is dihedral is simple enough to be carried out by hand; we have included the computation of the $h_{x,y,z}$ in section 4. This leaves us only with the two groups H_3 and H_4, and since the former is contained in the latter, the only case we need to consider is the Coxeter group of type H_4, of order 14400, with Coxeter diagram



Despite its rather modest size, the group H_4 poses a redoubtable computational challenge, even for present-day computers. It is all the more remarkable that property P_1 was checked already in 1987 by Dean Alvis [1] by explicitly computing all the Kazhdan-Lusztig polynomials. Quite a feat with the hardware of the time! Unfortunately, Alvis's programs have never been made available; to my knowledge, the only publicly available computer program capable of carrying out this computation is my own program `Coxeter` [2], which does it in about one minute on a modern-day personal computer. This still leaves open properties P_2 and P_3; the main purpose of this paper is to report on a computation, carried out as one of the first applications of version 3 of `Coxeter`, by which we prove :

1.3 Theorem. — *Properties P_2 and P_3 hold for the Coxeter group of type H_4.*

1.4 Corollary. — *The fifteen conjectures labelled* ***P1–P15*** *in Chapter 14 of Lusztig [8] hold for the Hecke algebra of the Coxeter group of type H_4, and in fact for the equal parameter Hecke algebra of any finite Coxeter group.*

Proof. — Lusztig shows in Chapter 15 of [8] that in the equal parameter case (which is automatic in type H_4), the fifteen conjectures all follow from P_1 and P_3. In view of our earlier remarks on the reduction to the irreducible case, and of the dihedral computation in section 4, the same argument can be applied to any finite Coxeter group.

2 Methodology

2.1. The verification of P_1 and P_2 is straightforward : one simply runs through the computation of the $P_{x,y}$ for all $x \leq y$ in the group. In fact, from the well-known property $P_{x,y} = P_{sx,y}$ whenever $sx > x$, $sy < y$ (cf. [5] (2.3.g)), and the analogous property on the right, for P_1 it suffices to consider the cases where $\mathrm{LR}(x) \supset \mathrm{LR}(y)$, where we denote $L(x) = \{s \in S \mid sx < x\}$ (resp. $R(x) = \{s \in S \mid xs < x\}$) the left (resp. right) descent set of x, and $LR(x) = L(x) \coprod R(x) \subset S \coprod S$; we call such pairs (x, y) *extremal pairs.* Taking also into account the symmetry $P_{x,y} = P_{x^{-1},y^{-1}}$, there are 2 348 942 cases to consider, which are easily tabulated by the program. The non-negativity of the polynomials is checked as they are found. For P_2, one also easily reduces to extremal pairs (x, y) and (z, y). The tough computation is for P_3; here there are *a priori* $14\,400^3 = 2\,985\,984\,000\,000$ (almost three trillion!) polynomials $h_{x,y,z}$ to be computed, and the only obvious symmetry is $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$ (but, as explained below, we don't even use that.)

2.2. The algorithm used in the computation is simple. For a fixed y, we compute the various $c_x.c_y$ by induction on the length of x, starting with $c_e.c_y = c_y$, where e denotes the identity element of W. To carry out the induction, we choose any generator $s \in S$ such that $sx < x$, and write :

$$c_x = c_s.c_{sx} - \sum_{\substack{z<sx \\ sz<z}} \mu(z, sx)c_z \tag{1}$$

where as usual $\mu(x, y)$ denotes the coefficient of v^{-1} in $p_{x,y}$ (which is also the coefficient of degree $\frac{1}{2}(l(y) - l(x) - 1)$ in $P_{x,y}$, and in particular is zero when the length difference of x and y is even.)

Then we may assume that $c_{sx}.c_y$ is already known, and similarly for the various $c_z.c_y$, so we are reduced to multiplications of the form $c_s.c_u$, for $s \in S$ and $u \in W$. When $su > u$ this is read off from formula (1) with $x = su$; and when $su < u$ one simply has $c_s.c_u = (v + v^{-1})c_u$ (see for instance [5]).

The information that is required for this computation is encoded in the W-graph of the group; once this graph is known, everything else just involves elementary operations on polynomials. Actually, it is obvious that the $h_{x,y,z}$ are in fact polynomials in $(v+v^{-1})$, so they are determined by their positive-degree part; it is this part that we compute and keep in memory. The only complication arises from the fact that we need to be careful about integer overflow; it turns out in fact that for H_4 all the coefficients of the $h_{x,y,z}$ fit into a 32-bit unsigned (and even signed) integer, but not by all that much: the largest coefficient that occurs is 710 904 968, which is only about six times smaller than 2^{32} = 4 294 967 296.

2.3. From the computational standpoint, this procedure has a number of desirable features. First and foremost, once the W-graph of the group has been determined, the problem splits up into 14400 independent computations, one for each y, so we can forget about the computation for a given y when passing to the next (this would not be true if we tried to use the symmetry $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$.) This is advantageous in terms of memory usage, could be used to parallelize the computation if necessary (it turns out that it hasn't been), and also means that the computation can be harmlessly interrupted (either voluntarily or involuntarily), at least if its progress is recorded somewhere : basically, the only penalty to pay for picking up an interrupted computation is the recomputation of the W-graph, which takes about half a minute. This is very valuable for computations running over several days, where there is always the risk of a system crash or power failure.

On the other hand, it is essential that for a fixed y, the full table of $h_{x,y,z}$ be stored in memory. In practice, there are many repetitions among these polynomials; so we store them in the form of a table of $14400^2 = 207\,360\,000$ pointers. Initially, this is the main memory requirement of the program; it is interesting to note that the cost doubles, from about 800 MB to about 1.6 GB, when we pass from a 32-bit to a 64-bit computer. It turns out that for a fixed y, the additional memory required to store the actual polynomials is small, and never exceeds 300 MB. So the full computation runs confortably in 2 GB of memory, and barely exceeds 1 GB on a 32-bit machine.

It is much more difficult to try to write down a full table of all the polynomials that occur as $h_{x,y,z}$. I have done this a number of times, but with the memory available on the machines to which I have had access, it has been necessary to split up the computation in about a hundred pieces to avoid memory overflow in the polynomial store, to keep the corresponding files in compressed form to avoid overflowing the hard disk, and then to merge those one hundred compressed files into a single compressed file, eliminating repetitions. At some point I have needed to store about 30 GB of compressed files—not something administrators are very happy about! In view of these difficulties, I have chosen not to make that version of the program available for the time being.

2.4. The computation has been done a number of times (writing files of all the distinct polynomials): at the Ecole Polytechnique, Centre de Calcul Médicis, Laboratoire STIX, FRE 2341 CNRS, on several computers, including a Compaq Alpha EV68 and an AMD Opteron server with 4 and 8 GB of main memory respectively, where it has taken about 5 days of CPU time, and twice at the Université Lyon I, Maply, UMR 5585 CNRS, on a Xeon processor with 4 GB of main memory, where it took 80 hours (not counting the final file merging pass, which took another 80 hours or so.) The program as presented here has run on our 2.7 GHz AMD Opteron server at the Institut Camille Jordan, using less than 2 GB of memory, in a little under 85 hours.

On the technological side, it seems that the time was just right for this computation : it makes full use of the 3Ghz processors, at least 2GB central memories, and 100+ GB hard disks that are found on typical low-end servers today. It would still be beyond the grasp of most present-day personal computers, however, although that, too, is changing fast!

3 Verification

3.1. Let's come now to the thorny issue of verification : what is the amount of trust that can be put in a result like this ? An obvious prerequisite is the availablility of the source code of the program that carries out the computation; this may be downloaded at `http://igd.univ-lyon1.fr/~ducloux/coxeter/coxeter3/positivity`.

This is actually the source code of an especially modified version of `Coxeter3`. All the extra code is contained in the file `special.cpp`; all the other files are identical to the ones in `Coxeter3`.

3.2. In addition to those already available in `Coxeter3`, the following commands are defined :

- `klplist` : prints out a list of all the distinct Kazhdan-Lusztig polynomials which occur in the group (so that in particular, one may re-check property P_1, although this was done already by the computer in the course of the computation);
- `decrklpol` : checks property P_2;
- `positivity` : checks the non-negativity of the $h_{x,y,z}$;
- `cycltable` : prints out a table of the $c_x.c_y$ for a fixed y;
- `cprod` : prints out a single product $c_x.c_y$;

For type H_4, on a decent server, the `cycltable` command should not take more than two minutes; `cprod` should usually take less than a minute (Note that the first call will take longer than subsequent ones, because the W-graph must be computed the first time.) So these commands give local access to the multiplication table of the Hecke algebra, thus opening up the "black box" a little.

The `positivity` command for type H_4 can also be executed through the little stand-alone command `coxbatch` that I have included; this will run the computation in the background. It writes any errors in `error_log`, and records the progress of the computation in `positivity_log`. After a successful run, `error_log` should be empty, and `positivity_log` should end with the line :

```
14399:  maxcoeff = 710904968
```

(elements of the group are numbered from 0 to 14399.)

3.3. As was explained in 2.2, the computation is entirely elementary once the W-graph of the group is known. The trust that one may place in this ingredient should be rather high, in my opinion, as it is computed with an algorithm which in simpler cases has been checked against other programs, and which even for H_4 has been checked against other algorithms for the same computation. (The latter check is perhaps more convincing than the former, for it may very well happen that some of the nastiest configurations occur for the first time in type H_4, or even only in type H_4, as it is such an exceptional group.)

For the actual non-negativity check, we are as far as I know in entirely uncharted territory. However, the code for the computation of the $h_{x,y,z}$ from the W-graph is really quite simple, so it can rather convincingly be checked by inspection. Another check is as follows : the order in which the computations are performed depends on the choice of a descent generator for x. By default, we always choose the first such generator in the internal numbering of `Coxeter`; however, it is easy enough to replace this by other "descent strategies" (for instance, choosing the first generator in some other ordering.) This will lead to a very different flow of recursion. I have done this replacing the first descent generator by the last one, and obtained the same output file, which is as it should be.

4 The case of dihedral groups

4.1. Let us now show how the algorithm described in 2.2 can be carried out by hand in the case where W is dihedral. This is not difficult and may well be known to experts, but I haven't been able to find the results in the literature. In Chapter 17 of [8], Lusztig computes the $h_{x,y,z}$ for an infinite dihedral group with *unequal* parameters. I have not been able to determine if his formulas can be specialized to yield the statement in Proposition 4.4 (although of course our computations for the infinte case are very similar to (and much simpler than) those in [8].)

4.2. Assume first that W is infinite dihedral. Let $S = \{s_1, s_2\}$, and for each $i \geq 0$ denote

$$[1,2,i\rangle = s_1 s_2 \dots \qquad [2,1,i\rangle = s_2 s_1 \dots$$
$$\langle i,1,2] = \dots s_1 s_2 \qquad \langle i,2,1] = \dots s_2 s_1$$

where in each case there are i terms in the product. When we need indeterminate generators, we will also use notation such as $[s,t,i\rangle$ for $s \neq t$ in S.

Denote for simplicity $c_1 = c_{s_1}$, $c_2 = c_{s_2}$. It is well-known that for dihedral groups all the Kazhdan-Lusztig polynomials $P_{x,y}$ are equal to 1 for $x \leq y$; incidentally, this proves that P_1 and P_2 are trivially verified. It follows that formula 2.2 (1) reduces to

$$\begin{aligned} &c_1.c_2 = c_{s_1.s_2} = c_{[1,2,2>} \\ &c_1.c_{[2,1,i>} = c_{[1,2,i+1>} + c_{[1,2,i-1>} \quad \text{for } i > 1 \end{aligned}$$

and of course $c_1.c_{[1,2,i>} = (v + v^{-1})\ c_{[1,2,i>}$ for $i > 0$, and similar formulæ for the left multiplication by c_2.

4.3. Now fix an element of the Kazhdan-Lusztig basis, that without loss of generality we may assume to be of the form $c_{<k,1,2]}$, with $k > 0$. Let s be the first generator in $<1, 2, k]$, so that $c_{<k,1,2]} = c_{[s,t,k>}$, and let $t \neq s$ be the other element of S. We wish to compute the various $c_{<i,s,t]}.c_{<k,1,2]}$, $c_{<i,t,s]}.c_{<k,1,2]}$, as i varies. Note that since the $c_{<j,1,2]}$, $j > 0$, form a basis of a left cell representation of $\mathcal{H}$, it is *a priori* clear that these products will be A-linear combinations of the $c_{<j,1,2]}$. Start with the $c_{<i,s,t]}.c_{<k,1,2]}$, where we may assume $i > 0$. For the first two values of i we get

$$\begin{aligned} &c_{<1,s,t]}.c_{<k,1,2]} = c_t.c_{[s,t,k>} = c_{[t,s,k-1>} + c_{[t,s,k+1>} = c_{<k-1,1,2]} + c_{<k+1,1,2]} \\ &c_{<2,s,t]}.c_{<k,1,2]} = c_s.c_t.c_{<k,1,2]} = c_{<k-2,1,2]} + 2c_{<k,1,2]} + c_{<k+2,1,2]} \end{aligned}$$

when $k > 2$, and

$$\begin{cases} c_t.c_{<1,1,2]} = c_{<2,1,2]} \\ c_s.c_t.c_2 = c_{<1,1,2]} + c_{<3,1,2]} \end{cases} \quad \text{for } k = 1$$

$$\begin{cases} c_t.c_{<2,1,2]} = c_{<1,2,1]} + c_{<3,2,1]} \\ c_s.c_t.c_{<2,1,2]} = 2c_{<2,2,1]} + c_{<4,2,1]} \end{cases} \quad \text{for } k = 2$$

Now applying the procedure from 2.2 yields the recursion formula:

$$c_{<i,s,t]}.c_{<k,1,2]} = c_r.c_{<i-1,s,t]}.c_{<k,1,2]} - c_{<i-2,s,t]}.c_{<k,1,2]} \quad \text{for } i > 2$$

where $r \in \{s, t\}$ is the first term in $c_{<i,s,t]}$. It follows easily that the non-zero terms in $c_{<i,s,t]}.c_{<k,1,2]}$ all correspond to indices j of the same parity, which changes when we go from one i to the next. Consequently, all these terms have a first generator *not* equal to r. If we write the coefficients in rows, we see that the coefficient at position j in row $i > 2$ is obtained by adding the coefficients at positions $j - 1$ and $j + 1$ in row $i - 1$, and subtracting the coefficient at position j in row $i-2$. For example, when $k=3$, the table looks like this (with dots indicating zeroes) :

	$k-2$	$k-1$	k	$k+1$	$k+2$	$k+3$	$k+4$	$k+5$
$i=1$	·	1	·	1	·	·	·	·
$i=2$	1	·	2	·	1	·	·	·
$i=3$	·	2	·	2	·	1	·	·
$i=4$	1	·	2	·	2	·	1	
$i=5$	·	1	·	2	·	2	·	1

Due to the exceptional multiplication rules $c_1.c_2 = c_{<2,2,1]}$ and $c_2.c_1 = c_{<2,1,2]}$, the first 1 that would appear in column 0 should be omitted (this occurs for $i=k$); another way of stating this is that we should run the algorithm in the half-plane $j>0$, and omit undefined terms. In this form, the procedure is valid for all values of $k>0$. Note that for $i>k$ the rows have constant length $k+1$, the next one being just the previous one shifted one unit to the right.

The case of the $c_{<i,t,s]}.c_{<k,1,2]}$ is similar but simpler; it appears as Proposition 7.7 *(a)* in Chapter 7 of [8]. Here all the non-zero coefficients are equal to $v+v^{-1}$, and we have a second order recursion as previously. Again the coefficients start expanding pyramidally, there is a "reflection" at column 0, and then we get a band of constant width k.

The final result may be stated as follows:

4.4 Proposition. — *Let W be an infinite dihedral group, and adopt the notation from 4.2 Let $k>0$, and let s be the first generator in $c_{<k,1,2]}$ (i.e., $s=1$ if k is even, $s=2$ if k is odd.) Let t be the other element in S. Then we have :*

$$c_{<i,s,t]}.c_{<k,1,2]} =$$
$$\begin{cases} c_{<k-i,1,2]} + 2c_{<k-i+2,1,2]} + \ldots + 2c_{<k+i-2,1,2]} + c_{<k+i,1,2]} & (0<i<k) \\ 2c_{<2,1,2]} + \ldots + 2c_{<2k-2,1,2]} + c_{<2k,1,2]} & (i=k) \\ c_{<i-k,1,2]} + 2c_{<i-k+2,1,2]} + \ldots + 2c_{<i+k-2,1,2]} + c_{<i+k,1,2]} & (i>k) \end{cases}$$

(where for $k=1$ the entry for $i=k$ should be interpreted as $c_{<2,1,2]}$), and similarly

$$c_{<i,t,s]}.c_{<k,1,2]} =$$
$$\begin{cases} (v+v^{-1})\, c_{<k-i+1,1,2]} + \ldots + (v+v^{-1})\, c_{<k+i-1,1,2]} & (0<i<=k) \\ (v+v^{-1})\, c_{<i-k+1,1,2]} + \ldots + (v+v^{-1})\, c_{<i+k-1,1,2]} & (i>=k) \end{cases}$$

with of course similar formulæ when $c_{<k,1,2]}$ is replaced by $c_{<k,2,1]}$

4.5. Now consider the case where W is a finite dihedral group of order $2m$, $m \geq 2$. Then of course $c_{<i,1,2]}$ is defined only for $i \leq m$, and moreover the action of c_1 and c_2 on $c_{<m,1,2]} = c_{<m,2,1]}$ is given by

$$c_1.c_{<m,1,2]} = c_2.c_{<m,1,2]} = (v+v^{-1})c_{<m,1,2]}$$

The consequence is that the recursive pattern described above gets modified starting from $i = m+1-k$. For that row, the expression $c_{<m-1,1,2]} + c_{<m+1,1,2]}$ that would have been obtained by applying the appropriate c_s to $c_{<m,1,2]}$ should be replaced by $(v+v^{-1})c_{<m,1,2]}$. The net effect is that the algorithm splits in two independent parts: one one hand we run the same algorithm as for the infinite case, but this time within the strip $0 < j < m$; on the other hand, starting from $i = m-k$, we add a term of the form $a_i c_{<m,1,2]}$, with $a_{m-k} = 1$, $a_{m-k+1} = v+v^{-1}$, $a_{m-k+2} = v^2+2+v^{-2}$, and

$$a_i = (v+v^{-1})a_{1-1} - a_{i-2} = v^d + 2v^{d-2} + \ldots + 2v^{-d+2} + v^{-d}$$

for $d = k+i-m > 2$. The procedure goes on until $i = m$, at which point the row from the first algorithm has disappeared altogether, and only the multiple of $c_{<m,1,2]}$ is left. For example, when $m = 9$, $k = 6$, and the $c_{<i,1,2]}.c_{<k,1,2]}$ (we have $s = s_1$ in this case), we get the following table for the first part of the algorithm :

	$k-5$	$k-4$	$k-3$	$k-2$	$k-1$	k	$k+1$	$k+2$
$i=1$	·	·	·	·	1	·	1	·
$i=2$	·	·	·	1	·	2	·	1
$i=3$	·	·	1	·	2	·	2	·
$i=4$	·	1	·	2	·	2	·	1
$i=5$	1	·	2	·	2	·	1	·
$i=6$	·	2	·	2	·	1	·	·
$i=7$	1	·	2	·	1	·	·	·
$i=8$	·	1	·	1	·	·	·	·

and row 9 is zero. The obvious symmetry in the shape is a general fact.

Note that if we look at the $\mathbf{Z}$-basis of $\mathcal{H}$ afforded by the $v^d c_w$, $d \in \mathbf{Z}$, and give $v^d c_{<j,1,2]}$ degree $j+d$, then the sum of the coefficients of $c_{<i,s,t]}.c_{<k,1,2]}$ in each degree is the same for the finite and the infinite cases; whatever is missing from the infinite picture is exactly reflected in the coefficient of $c_{<m,1,2]}$. So a concise statement of the result is as follows :

4.6 Proposition. — *Let W be dihedral of order $2m$, $m \geq 2$. Then the formulæ from Proposition 4.4 remain valid, except that one must have $i \leq m$, and that any expression of the form*

$$c_{<m+d,1,2]} + c_{<m-d,1,2]} \qquad d > 0$$

that can be taken out of the formula should be replaced by $(v^d + v^{-d})c_{<m,1,2]}$.

4.7 Example. — Pursuing the earlier example where $m = 9$ and $k = 6$, and taking for instance the case where $i = 6$, putting together the sixth row in the above table and the expression for the coefficient a_i, which corresponds to $d = 3$, we get:

$$c_{<6,1,2]} . c_{<6,1,2]} = \\ 2c_{<2,1,2]} + 2c_{<4,1,2]} + c_{<6,1,2]} + (v^3 + 2v + 2v^{-1} + v^{-3})\, c_{<9,1,2]} \quad (1)$$

The corresponding expression for the infinite group would be

$$2c_{<2,1,2]} + 2c_{<4,1,2]} + 2c_{<6,1,2]} + 2c_{<8,1,2]} + 2c_{<10,1,2]} + c_{<12,1,2]}$$

from which we recover (1) by taking out out one copy of $c_{<6,1,2]} + c_{<12,1,2]}$ and two copies of $c_{<8,1,2]} + c_{<10,1,2]}$.

It is easy to get many other examples from the program — of course playing with the program is how the above statements were found in the first place.

5 Questions

5.1. On looking at the lists of polynomials which occur as $h_{x,y,z}$, one immediately notices that they are not only non-negative, but have a much stronger positivity property : if we denote d the degree of $h_{x,y,z}$, then $v^d h_{x,y,z}$ is a polynomial in $q = v^2$, which is *unimodal* (recall that this means that the coefficients increase up to a point, which in this case has to be the middle because of the symmetry of the $h_{x,y,z}$, and decrease from there.) In the course of the computation, the program checks unimodality for all $h_{x,y,z}$, and prints an error message on `error_log` in case of failure. Hence unimodality holds for the Hecke algebra of type H_4.

5.2. For Weyl groups, there is one case where it is easy to prove that unimodality holds : *viz.* the case where $y = w_0$ is the longest element in the group. From the properties of the $\leq_{\mathrm{LR}}$ preorder it is clear that $A.c_{w_0}$ is a two-sided ideal in $\mathcal{H}$; so we may write $c_x.c_{w_0} = h_x c_{w_0}$, where $h_x = h_{x,w_0,w_0}$. Now it is clear that $t_s.c_{w_0} = v.c_{w_0}$ for all s, hence $t_x.c_{w_0} = v^{l(x)}c_{w_0}$, and

$$c_x.c_{w_0} = \sum_{z \leq x} p_{z,x} v^{l(z)} c_{w_0}$$

from which it follows immediately, using the expression of the intersection homology Poincaré polynomial in terms of Kazhdan-Lusztig polynomials ([6] Theorem 4.3.) that $v^{l(x)}h_x$ is equal to the Poincaré polynomial of the intersection (hyper)cohomology of the Schubert variety X_x. The unimodality then follows from the so-called hard Lefschetz theorem. As far as I know, the unimodality property for general $h_{x,y,z}$ is an open question, even for Weyl groups.

5.3. Clearly, all the results about the Hecke algebra of type H_4 which are stated in this paper point to the fact that there is a hidden geometry here that is begging to be discovered. Hopefully, the facts about this geometry which the program opens up will help us understand what is going on, and serve as a guide towards the solution. I should be very happy if this turns out to be the case.

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